Problem Set II: Due Thursday, January 29, 2015
                    3 **Problems will be graded.**

1.)   Problem 6 from Set I.

        Consider a free nonlinear oscillator which satisfied the equation

$$\ddot{x} + \omega_0^2 x = -\alpha x^2 - \beta x^3 .$$

        Use Poincare-Linstedt (reductive) perturbation theory to calculate the non-linear
        frequency shift and lowest order non-trivial solution.

2.)   On WKB: Consider the differential equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{Q(x)}{\varepsilon^2}\psi = 0 .$$

a.)   Use an ansatz of the form $\psi \cong \exp\left[\dfrac{i}{\varepsilon} \sum\limits_{n=0}^{\infty} \varepsilon^n \phi_n(x)\right]$ to derive coupled equations

        for ϕ_0 , ϕ_1 .

b.)   How does the ϕ_0 equation compare to the eikonal equation?

c.)   Solve for ϕ_0 and ϕ_1 , and use them to write an approximate form of ψ . Discuss
        your result.

d.)   What are the conditions of validity for the approximation strategy used here?

3.)   Consider an acoustic wave propagating in a 3D medium with index of refraction
        $n = n(x,y,z)$.

a.)   For a short wavelength, constant frequency excitation, so that ray theory is
        applicable, discuss under what circumstances you will be able to solve the ray
        equations.

b.)   In the case of a.), calculate the eikonal phase $\phi(\underline{x})$.

4.)   Determine the variation with altitude of the amplitude of a sound wave propagating in an isothermal atmosphere with gravity.

5.)   Derive an expression for the relationship between the unit normal vector to an acoustic path and the profile of the index of refraction. Relate this result to its counterpart for particle motion, using the equation for a particle path.

6a.)   What are the general conditions on the potential $V(r,\phi,z)$ for separability of the Hamilton-Jacobi equation for particle motion in cylindrical coordinates?

b.)   Solve the Hamilton-Jacobi equation by separation, assuming $V(r,\phi,z)$ has the requisite form.

7a.)   Consider the linearized acoustic equations

$$\frac{\partial \tilde{\rho}}{\partial t} + \underline{v} \cdot \nabla \tilde{\rho} = -\rho_0 \underline{\nabla} \cdot \underline{\tilde{v}}$$

$$\rho_0 \left(\frac{\partial \underline{\tilde{v}}}{\partial t} + \underline{v} \cdot \nabla \underline{\tilde{v}} \right) = -c_s^2 \nabla \tilde{\rho}$$

for an acoustic wave in a *flowing* medium, with flow velocity $\underline{v}$. This means that the frequency is Doppler shifted. Neglecting the spatial variation of the ambient flow in comparison to the wave length, derive the eikonal equation

$$\left(\frac{\partial \phi}{\partial t} + \underline{v} \cdot \underline{\nabla}\phi \right)^2 = \left(\underline{\nabla}\phi \right)^2 c_s^2(\underline{x}).$$

Show this is equivalent to the predictable:

$$\left(\omega - \underline{k} \cdot \underline{v} \right)^2 = k^2 c_s^2(\underline{x}).$$

b.)   Now, derive the ray equations with flow.

c.)   Explain the physics of the ray equation $\dfrac{d\underline{k}}{\partial t} = -\dfrac{\partial}{\partial x}(\omega + \underline{k} \cdot \underline{v})$. Use this to explain how a vertically sheared horizontal affects acoustic propagation. Why is it difficult to hear shouts from a distance on a windy day?

d.)   Show that the ray equations with flow are Hamiltonian.

8.)   Consider the nonlinear Klein-Gordon equation, a simple scalar field theory.

$$\frac{1}{c_0^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi + \alpha \phi^3 = 0 \ .$$

a.)   If one looks for propagating solutions of the form $\phi(x,t) = \phi(x - ct)$, show that the linear problem reduces to

$$\left(c^2 - c_0^2\right)\phi'' + m^2 c_0^2 \phi = 0$$

so $c^2 = c_0^2 + m^2/k^2$ gives the speed. Here k is a wave number.

b.)   What is the tricky point of the nonlinear problem? How is it best confronted?

c.)   Use reductive perturbation theory with:

$$c = c^{(0)} + a^2 c^{(2)} + \dots$$

$$\phi = a\phi_1 + a^3 \phi_3 + \dots$$

to show that nonlinearity induces a speed shift, so:

$$c = \left(c_0^2 + \frac{m^2}{k^2}\right)^{1/2}\left[1 + \frac{3}{8}\frac{\alpha \, a^2}{\left(c_0^2 k^2 + m^2\right)}\right].$$

9.) Consider the forced Duffing Equation:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x + \beta x^3 = f_{ext}/m .$$

a.) Write an expression for the amplitude a when $\beta = 0$. Consider $\varepsilon = \omega\text{-}\omega_{res}$ finite but not large. Of course, $f_{ext} = f_{ext} \cos\omega t$.

b.) Now, what happens for $\beta \neq 0$? Write the expression for α .

c.) Draw the curve of a^2 vs ε for $\beta \neq 0$, for various values of f_{ext} .

d.) When can a^2 be double-valued? What does this mean?

e.) Calculate the critical value of f for double valuedness.